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Generalized Chebyshev Polynomials of Third Kind

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Abstract: In this paper, the generalized third-kind Chebyshev polynomials are considered. Motivated by the fact that the Bernstein polynomial basis is numerically stable, the generalized Chebyshev polynomials of third kind are characterized by writing them in terms of the Bernstein basis. This representation is given in explicit form. Moreover, the weighted definite integrals of the generalized third-kind Chebyshev polynomials multiplied with the Bernstein polynomials are found.

keywords: Basis Transformation; Third-Kind Generalized Chebyshev Polynomials; Bernstein Polynomials; Computer Aided Design.

MSC 2010: 41A17; 65D17; 65D30.

1 Introduction

Orthogonal polynomials play fundamental role in approximation theory. Functions are approximated using orthogonal polynomials easily and efficiently, see [3]. Koornwinder introduced in [11] orthogonal polynomials on the interval $[-1, 1]$ with weight function combined from the Jacobi weight function and two delta functions at the boundaries -1 and 1. These polynomials are called the generalized Jacobi polynomials. In [10], it is proved that they uniquely satisfy a differential equation, and the explicit representations for the coefficients are also given. In [2], the generalized Chebyshev polynomials of the second kind are characterized, and a closed form using the Bernstein basis is determined. In [9], important properties are derived for the Barnes' multiple Bernoulli and Hermite mixed-type polynomials. In [16], inequalities of Hermite-Hadamard type for functions of bounded second-order derivative are established, see also [4, 8, 11, 14].

2 Third-Kind Chebyshev Polynomials

The classical third-kind Chebyshev polynomials, $V_n(x)$, are orthogonal on $[-1,1]$ with respect to the weight function $w(x)=\sqrt{\frac{1+x}{1-x}}$. They have the trigonometric representation:

$$V_n(x)=\frac{\cos((n+\frac{1}{2})\cos^{-1}(x))}{\cos(\frac{1}{2}\cos^{-1}(x))},\quad \forall x\in\ [-1,1]. \tag{1}$$

The Chebyshev polynomials of third kind satisfy the following linear homogeneous differential equation of the second order:

$$(1-x^2)y^{''}+(1-2x)y^{'}+n(n+1)y=0.$$

For more, see [1, 17].

For $M,N\geq 0$, the generalized third-kind Chebyshev polynomials, $V_n^{(M,N)}(x)$, are the orthogonal polynomials on the interval [-1, 1] with respect to the weight function

$$w(x)=2(1-x)^{\frac{-1}{2}}(1+x)^{\frac{1}{2}}+M\delta(x+1)+N\delta(x-1),$$

where $\delta(x+1)$ is the Dirac delta function defined by the following property:

$$\delta(t)=\begin{cases}\infty, & \text{if } t=\ 0\\ 0, & \text{if } t\neq\ 0\end{cases}.$$

It satisfies the property, see [17]:

$$\int_{t_1}^{t_2}\delta(t)dt=1,\quad 0\in[t_1,t_2].$$

They satisfy the following differential equation of the form

$$M\ \sum_{i=0}^{\infty}a_i(x)\ y^{(i)}(x)+N\ \ \sum_{i=0}^{\infty}b_i(x)\ y^{(i)}(x)+MN\ \ \sum_{i=0}^{\infty}c_i(x)\ y^{(i)}(x)+(1-x^2)y^{''}(x)+$$

$$+(1-2x)\ y^{'}(x)+n(n+1)\ y(x)=0.$$

It is important to study these polynomials because they can be written in explicit form and consequently make important contributions to the field of approximation theory. Many formulas and applications can be found and applied. The final results are given in terms of Bernstein polynomials which are defined on the interval [0,1]; therefore, the interval of the third-kind Chebyshev polynomials is shifted by the map $x=2u-1$. Thus, the third-kind Chebyshev polynomials $V_n(u)$ of degree n on $[0,1]$ become the orthogonal polynomials on [0,1] with respect to the weight function: $w(u)=\sqrt{\frac{u}{1-u}}$.

3 Bernstein Basis

The n-th degree Bernstein polynomials on [0,1] are given by the formula:

$$B_i^n(u)=\left(\begin{array}{c}n\\ i\end{array}\right)(1-u)^{n-i}\ u^i,\quad u\in[0,1],\quad i=0,\ldots,n,$$

where $\begin{pmatrix} n \\ i \end{pmatrix} = \frac{n!}{i!\ (n-i)!}$.

The definite integrals of Bernstein polynomials are given by, see [12].

$$\int_0^1 B_i^n(u)\ du = \frac{1}{n+1}, \quad i = 0, 1, \ldots, n.$$

From the definition of the Bernstein polynomials it is clear that $B_i^n(u) \geq 0$, $\forall u \in [0,1]$ and $B_i^n(u) = B_{n-i}^n(1-u)$.

The Bernstein polynomials play an important role in the development of Bézier curves and surfaces in Computer Aided Geometric Design. They posses important geometric, analytic, and stability properties, see [6, 7]. A degree n Bézier curve is written in the following form:

$$p(t) = \sum_{i=0}^{n} p_i B_i^n(t) =: \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad 0 \leq t \leq 1. \tag{2}$$

The points $p_0, p_1, \ldots, p_n$ are called the control points, see [15].

4 Preliminaries

The factorial of an integer n is defined as follows:

$$n! = \begin{cases} n(n-1)(n-2)\cdots(2)(1), & n > 0 \\ 0 \qquad\qquad\qquad\qquad\qquad , & o.w \end{cases}$$

The double factorial of an integer n is defined by the following formula:

$$n!! = \begin{cases} n(n-2)\cdots(4)(2), & \text{if } n \text{ even} \\ n(n-2)\cdots(3)(1), & \text{if } n \text{ odd} \end{cases}. \tag{3}$$

For properties of the double factorial, see [12].

The beta function is defined as follows:

$$B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}du.$$

The Pochhamar symbol $(x)_n$ is defined as follows:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\ldots(x+n-1), \quad n \geq 0.$$

It is a notation used in the theory of special functions for the rising factorial, and it is also known as the rising factorial power or ascending factorial. In combinatorics, the notation $x^{(n)}$ or x^{n} is used for the rising factorial, while $x_{(n)}$ or x_n denotes the falling factorial. The first few terms of $(x)_n$ for non negative integers n are given by

$$(x)_0 = 1,$$

$$(x)_1 = x,$$

$$(x)_2 = x^2 + x,$$

$$(x)_3 = x^3 + 3x^2 + 2x,$$
$$(x)_4 = x^4 + 6x^3 + 11x^2 + 6x.$$

The Pochhammer symbol satisfies the following property:

$$(-x)_n = (-1)^n \ (x-n+1)_n, \quad n \geq 0.$$

Special values include

$$(1)_n = n!,$$
$$(\frac{1}{2})_n = \frac{(2n-1)!!}{2^n},$$

where !! denotes the double factorials.

5 Generalized Chebyshev Polynomials of Third Kind

The generalized Chebyshev polynomials of third kind, $V_n^{(M,N)}(x)$, $M, N \geq 0$, are generalizations of the Chebyshev polynomials of third kind. They can be written as combination of third-kind Chebyshev polynomials in the following form, see [11],

$$V_n^{(M,N)}(x) = \frac{(2n-1)!!}{(2n)!!} \ V_n(x) \ + \sum_{k=0}^{n} \lambda_k \ \frac{(2k-1)!!}{(2k)!!} \ V_k(x), \tag{4}$$

where

$$\lambda_k = Mq_k + Nr_k + MNs_k,$$

and

$$q_k = \frac{(\frac{5}{2})_{k-1}(2)_{k-1}}{(\frac{1}{2})_{k-1}(k-1)!}, \quad r_k = \frac{(2)_{k-1}}{(k-1)!}, \quad s_k = \frac{4(2)_{k-1} \ (2)_k}{3(k-2)!(k-1)!}. \tag{5}$$

The generalized third-kind Chebyshev polynomials $V_n^{(M,N)}(x)$ have the following property, see [11]:

$$V_n^{(M,N)}(1) = \frac{\left(\frac{1}{2}\right)_n}{(n)!} + \frac{\left(\frac{3}{2}\right)_n \ (2)_n \ n \ M}{\frac{3}{2} \ (n)!}.$$

They also satisfy the symmetric relation:

$$V_n^{(M,N)}(-x) = (-1)^n V_n^{(M,N)}(x).$$

6 Characterization in Bernstein Form

In this section, the generalized third-kind Chebyshev polynomials, $V_n^{(M,N)}(x)$, are characterized by writing them in terms of the Bernstein basis. This representation is motivated by the fact that the Bernstein polynomial basis is numerically stable, see [5]. Let $M, N \geq 0$, the generalized third-kind Chebyshev polynomials $V_n^{(M,N)}(x)$, see [10], can be written as

$$V_n^{(M,N)}(x) = \frac{(2n-1)!!}{(2n)!!} \ V_n(x) + MQ_n(x) + NR_n(x) + MNS_n(x), \quad n = 1, 2, \ldots, \tag{6}$$

where

$$Q_0(x) = R_0(x) = S_0(x) = 0$$

and for $n = 1, 2, 3, \ldots$

$$Q_n(x) = \frac{\left(\frac{5}{2}\right)_{n-1} \ (2)_{n-1} \ (2n-1)!!}{\left(\frac{1}{2}\right)_n \ n! \ (2n)!!} \left[n(n+1)V_n(x) - (\frac{3}{2})(x-1)DV_n(x)\right], \tag{7}$$

$$R_n(x) = \frac{\left(\frac{3}{2}\right)_{n-1} \ (2)_{n-1} \ (2n-1)!!}{n! \ (2n)!! \left(\frac{3}{2}\right)_n} \left[n(n+1)V_n(x) - (\frac{1}{2})(x+1)DV_n(x)\right], \tag{8}$$

and

$$S_n(x) = \frac{4(2n-1)!!(n+1)! \ (2)_{n-1} \ (2)_n}{3(2n)!! \ (n-1)! \ n!} \left[n(n+1)V_n(x) - \left[(\frac{3}{2})(x-1) + \frac{1}{2}(x+1)\right] DV_n(x)\right]. \tag{9}$$

We know that the generalized third-kind Chebyshev polynomials satisfy the symmetry relation which implies that $Q_n(x) = (-1)^n Q_n(-x), S_n(x) = (-1)^n S_n(-x)$ for $n = 1, 2, \ldots$. From (7) and (8) it follows that

$$Q_n(1) = \frac{\left(\frac{5}{2}\right)_{n-1} \ (2)_n \ (2n-1)!!}{\left(\frac{1}{2}\right)_n \ (n-1)! \ (2n)!!} V_n(1), \quad n = 1, 2, 3, \ldots \tag{10}$$

and

$$R_n(-1) = \frac{\left(\frac{3}{2}\right)_{n-1} \ (2)_n \ (2n-1)!!}{\left(\frac{3}{2}\right)_n \ (n-1)! \ (2n)!!} V_n(-1), \quad n = 1, 2, 3, \ldots \tag{11}$$

Note that the representations (7), (8), and (9) imply that for $n = 1, 2, 3, \ldots$, we have, see [10],

$$Q_n(x) = \sum_{k=0}^{n} q_k \frac{(2k-1)!!}{(2k)!!} V_k(x), \tag{12}$$

$$R_n(x) = \sum_{k=0}^{n} r_k \frac{(2k-1)!!}{(2k)!!} V_k(x), \tag{13}$$

$$S_n(x) = \sum_{k=0}^{n} \ s_k \frac{(2k-1)!!}{(2k)!!} V_k(x),$$

where q_k, r_k and s_k are defined by (5).

Since the final results are written in Bernstein polynomials which are defined on [0,1], then also $V_r^{(M,N)}(u)$ are shifted to be defined on [0,1].

In the following theorem the generalized third-kind Chebyshev polynomials, $V_r^{(M,N)}(u)$, of degree r are written in terms of the Bernstein polynomials $B_i^r(u), i = 0, 1, \ldots, r$ of degree r, $u \in [0, 1]$.

Theorem 1: For $M, N \geq 0$, the generalized third-kind Chebyshev polynomials $V_r^{(M,N)}(u)$ of degree r, $r = 0, 1, \ldots,$ have the following Bernstein representation:

$$V_r^{(M,N)}(u) = \frac{(2r-1)!!}{(2r)!!} \ \sum_{j=0}^{r} (-1)^{r-j} \vartheta_{j,r} B_j^r(u) +$$

$$\sum_{k=0}^{r} \lambda_k \frac{(2k-1)!!}{(2k)!!} \ \sum_{i=0}^{k} (-1)^{k-i} \vartheta_{i,k} \ B_i^k(u), \ \ u \in [0, 1],$$

where

$$\lambda_k = Mq_k + Nr_k + MNs_k \tag{14}$$

and

$$\vartheta_{0,k} = 1,\ \vartheta_{i,k} = \frac{\binom{2k+1}{2i}}{\binom{k}{i}}, \quad i = 1, 2, \ldots, k. \tag{15}$$

Proof: Our aim is to write the generalized third-kind Chebyshev polynomials $V_r^{(M,N)}(u)$ of degree r as a linear combination of the Bernstein polynomial basis $B_j^r(u), \quad j = 0, 1, \ldots, r$ of degree r in explicit form. The third-kind Chebyshev polynomial, $V_n(u)$, of degree n is expressed in the degree n Bernstein basis
$B_0^n(u), B_1^n(u), \ldots, B_n^n(u)$ as follows, see [14]:

$$V_n(u) = \sum_{k=0}^{n} \frac{(-1)^{n-k} \binom{2n+1}{2k} B_k^n(u)}{\binom{n}{k}}.$$

Substituting the last relation yields

$$V_r^{(M,N)}(u) = \frac{(2r-1)!!}{(2r)!!}\ V_r(u) + \sum_{k=0}^{r} \lambda_k\ \frac{(2k-1)!!}{(2k)!!}\ V_k(u).$$

This leads to the formula

$$V_r^{(M,N)}(u) = \frac{(2r-1)!!}{(2r)!!} \left[\sum_{j=0}^{r} \frac{(-1)^{r-j} \binom{2r+1}{2j}}{\binom{r}{j}} B_j^r(u) \right] +$$

$$\sum_{k=0}^{r} \lambda_k \frac{(2k-1)!!}{(2k)!!} \left[\sum_{i=0}^{k} \frac{(-1)^{k-i} \binom{2k+1}{2i}}{\binom{k}{i}} B_i^k(u) \right].$$

Doing some simplifications, we get

$$V_r^{(M,N)}(u) = \frac{(2r-1)!!}{(2r)!!} \sum_{j=0}^{r} (-1)^{r-j}\ \vartheta_{j,r}\ B_j^r(u) + \sum_{k=0}^{r}\ \lambda_k \frac{(2r-1)!!}{(2r)!!} \left[\sum_{i=0}^{k} (-1)^{k-i}\ \vartheta_{i,k} B_i^k(u) \right],$$

where λ_k and $\vartheta_{j,r}$ are defined in (14) and (15), respectively.

7 Weighted Definite Integral

In this section, the definite integral of weighted generalized third-kind Chebyshev polynomials multiplied with the Bernstein polynomials are given in explicit form in the following theorem.

Theorem 2: Let $B_r^n(u)$ be the Bernstein polynomial of degree n and $V_i^{(M,N)}(u)$ be the generalized third-kind Chebyshev polynomial of degree i; then for $i, r = 0, 1, \ldots, n$, we have

$$\int_0^1 u^{\frac{1}{2}}(1-u)^{\frac{-1}{2}}\ B_r^n(u)\ V_i^{(M,N)}(u)du$$

$$= \binom{n}{r} \frac{(2i-1)!!}{(2i)!!} \sum_{k=0}^{i} (-1)^{i-k} \binom{2k+1}{2i} B(r+k+\frac{3}{2}, n+i-r-k+\frac{1}{2})$$

$$+\sum_{d=0}^{i} \lambda_d \binom{n}{r} \frac{(2d-1)!!}{(2d)!!} \sum_{j=0}^{d} (-1)^{d-j} \binom{2d+1}{2j} B(r+j+\frac{3}{2}, n+d-r-j+\frac{1}{2}),$$

where $B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}du$ is the beta function.

Proof: Let

$$I = \int_0^1 u^{\frac{1}{2}}(1-u)^{\frac{-1}{2}}\ B_r^n(u)\ V_i^{(M,N)}(u)du.$$

Using the relation given in Theorem 1 leads to

$$I = \frac{(2i-1)!!}{(2i)!!} \int_0^1 u^{r+\frac{1}{2}}(1-u)^{n-r-\frac{1}{2}} \binom{n}{r} \sum_{k=0}^{i} \frac{(-1)^{i-k}\binom{2k+1}{2i}}{\binom{k}{i}} B_k^i(u)du$$

$$+\sum_{d=0}^{i} \lambda_d \frac{(2d-1)!!}{(2d)!!} \int_0^1 u^{r+\frac{1}{2}}\ (1-u)^{n-r-\frac{1}{2}} \binom{n}{r} \times \sum_{j=0}^{d} \frac{(-1)^{d-j}\binom{2d+1}{2j}}{\binom{d}{j}} B_j^d(u)du$$

$$= \frac{(2i-1)!!}{(2i)!!} \binom{n}{r} \sum_{k=0}^{i} \frac{(-1)^{i-k}\binom{2k+1}{2i}}{\binom{k}{i}} \binom{k}{i} \times \int_0^1 u^{r+k+\frac{1}{2}}\ (1-u)^{n+i-r-k-\frac{1}{2}}du$$

$$+\sum_{d=0}^{i} \lambda_d \frac{(2d-1)!!}{(2d)!!} \binom{n}{r} \sum_{j=0}^{d} \frac{(-1)^{d-j}\binom{2d+1}{2j}}{\binom{d}{j}} \binom{d}{j}$$

$$\times \int_0^1 u^{r+j+\frac{1}{2}}\quad (1-u)^{n+d-r-j-\frac{1}{2}}du$$

$$= \frac{(2i-1)!!}{(2i)!!}\begin{pmatrix} n \\ r \end{pmatrix}\sum_{k=0}^{i}(-1)^{i-k}\begin{pmatrix} 2k+1 \\ 2i \end{pmatrix}\times\int_0^1 u^{r+k+\frac{1}{2}}\,(1-u)^{n+i-r-k-\frac{1}{2}}du$$

$$+\sum_{d=0}^{i}\lambda_d\,\frac{(2d-1)!!}{(2d)!!}\begin{pmatrix} n \\ r \end{pmatrix}\sum_{j=0}^{d}(-1)^{d-j}\begin{pmatrix} 2d+1 \\ 2j \end{pmatrix}\times\int_0^1 u^{r+j+\frac{1}{2}}\quad(1-u)^{n+d-r-j-\frac{1}{2}}du.$$

Substituting the beta function completes the proof.

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